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LOAD DISTRIBUTION OF MULTI-FASTENER LAMINATED COMPOSITE JOINTS

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Abstract—In this paper, a general and effective method is developed for load distribution of multi-fastener laminated composite joints by using the Faber series expansion (Cutis, 1971, *Am. Math. Monthly* **78**, 577–596; Kosmodamianskii and Chernick, 1981, *Appl. Mech.* **17**(6), 94–100) and a complex potential method (Lekhnitskii, 1957, *Anisotropic Plates*, Gordon and Breach Science Publishers, New York; Muskhelishvili, 1966, *Some Basic Problems of the Mathematical Theory of Elasticity* (5th Edn), Nauka, Moscow) in the plane theory of elasticity. As an application, the effects of friction and fitting tolerance on load distribution of a four-fastener joint are studied. The method presented here can be generalized to solve other problems in this field.

INTRODUCTION

From a practical point of view, a mechanically fastened joint of a structure would involve multiple hole/pin connections in which there exists, in general, some positive fitting tolerances. Considering the re-distribution of loads acting on hole/pin connections due to the plastic deformation of ductile materials, the calculation of the ultimate load of a joint of metal structures would not be affected by the existence of a small positive fitting tolerance, and sometimes we can simply assume that the subjecting load of a joint is uniformly distributed to every hole/pin connection.

However, for a joint in a fiber/resin laminated composite structure, we must consider different aspects from those considered for the brittleness of the material; all of the hole/pin connections are not in contact with each other simultaneously for the existence of a positive fitting tolerance, and then the strength of the joint would be decreased, especially in applying an alternating load. As far as we know, no-one has considered the effects of a positive fitting tolerance.

Fan and his students have discussed some problems of the strength of a laminated composite joint (Fan, 1987; Fan and Wu, 1988, 1989; Wu and Fan, 1988; Xu and Fan, 1991). This paper deals with the load distribution of multi-fastener joints, especially in considering the effects of a positive fitting tolerance.

ANALYSIS

Consider a laminated composite joint having four hole/pin connections in series, with equal centre-to-centre distance L and equal diameter D, and being subject to a tensile load P along the hole-centre line, shown in Fig. 1. Its upper plate A and lower plate B can be considered as two homogeneous, orthotropic, infinite thin plates whose material directions

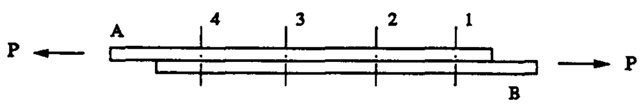

Fig. 1. A laminated composite joint having four hole/pin connections.

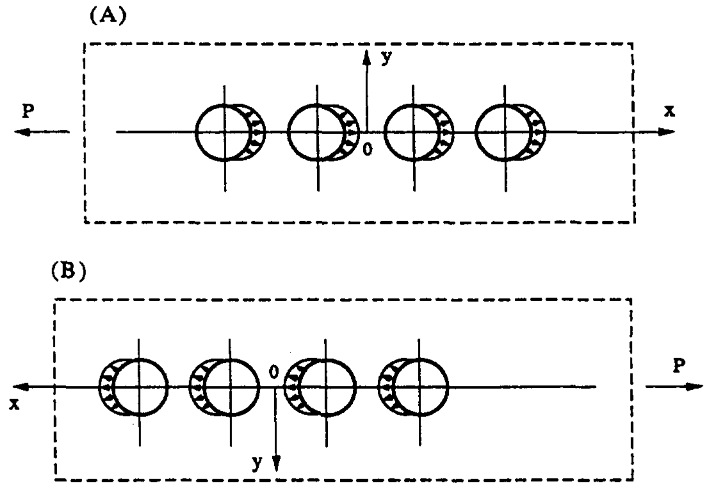

Fig. 2. The computational model of the joint.

are assumed to coincide with the co-ordinate axes, shown in Fig. 2. It possesses a symmetry in its geometry, loading and elastic tensor, and therefore a symmetry in its stress field about the x-axis. The contacting stresses between every hole/pin connection are then even functions of θ (Fig. 3) and can be expanded in a cosine series. For simplicity, the radial pressure for Hole j can be assumed to be (Fig. 3)

$$p_j(\theta) = \begin{cases} p_{j0}\cos\theta & -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}, \\ 0 & \frac{\pi}{2} \leqslant \theta \leqslant \frac{3}{2}\pi, \end{cases} \tag{1}$$

and

$$p_{j0} = \frac{4P_j}{\pi D h} \quad (j = 1, 2, 3, 4), \tag{2}$$

where p_{j0} is the peak stress for Hole j, P_j is the transferred load or the resultant force of the contacting stresses for Hole j and h is the plate thickness. This assumption is too basic for strength analysis. However the load distribution of multi-fastener joints is mainly controlled by the relative displacements between hole-centres. In a limiting case, as the upper and lower plates approach rigidity, all of the relative displacements between hole-centres approach zero and the external load is uniformly distributed to every hole/pin

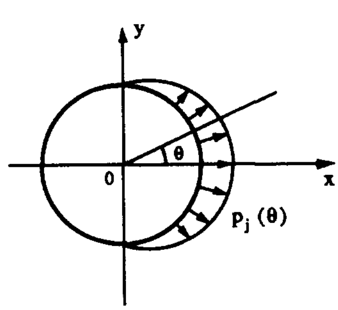

Fig. 3. The distribution of radial pressure along a hole contour.

connection. Thus the above assumption, eqn (1), is satisfactory by Saint-Venant's principle when the centre-to-centre distance is relatively large compared to the hole diameter. Once the transferred load of every hole has been determined, we can make the more rigorous assumption of the contacting stress distribution for strength analysis.

The distribution of tangential (friction) stress $q_j(\theta)$ is assumed to be as in Fig. 4. Its magnitude

$$|q_j(\theta)| = fp_j(\theta) \quad (j = 1, 2, 3, 4),$$

where the friction coefficient f is between 0 and 1. Now if the whole contact area is assumed to be a slip region, then the friction force will change sign discontinuously at the point where the relative displacement of the edge with respect to the fastener changes sign. It has no physical relevance (de Jong, 1983). However we believe that this assumption has a small effect on the load distribution when the centre-to-centre distances are relatively large compared to the hole diameter of a joint.

For the case of four equal holes in series the two complex potential functions (Fan and Wu, 1988; Lekhnitskii, 1957; Xu and Fan, 1991) for the upper plate A are

$$\begin{cases} \phi(z_1) = \sum_{j=1}^{4} A_j \ln(z_1 - b_j) + \phi_0 + \sum_{j=1}^{4} \sum_{n=1}^{\infty} \phi_{jn}[w_1(z_1 - b_j)]^{-n}, \\ \psi(z_2) = \sum_{j=1}^{4} B_j \ln(z_2 - b_j) + \sum_{j=1}^{4} \sum_{n=1}^{\infty} \psi_{jn}[w_2(z_2 - b_j)]^{-n}, \end{cases} \tag{3}$$

where real numbers b_j $(j = 1, 2, 3, 4)$ are complex coordinates in z, z_1, z_2 planes of four hole centres for they lie on the real axis; ϕ_0, ϕ_{jn} and ψ_{jn} $(j = 1, 2, 3, 4;\ n = 1, 2, 3, \ldots)$ are coefficients of the respective series. Complex constants A_j, B_j and their complex conjugates $\bar{A}_j$, $\bar{B}_j$ $(j = 1, 2, 3, 4)$ are determined from the following equations:

$$\begin{cases} A_j + B_j - \bar{A}_j - \bar{B}_j = \dfrac{P_{jy}}{2\pi hi}, \\ s_1 A_j + s_2 B_j - \bar{s}_1 \bar{A}_j - \bar{s}_2 \bar{B}_j = -\dfrac{P_{jx}}{2\pi hi}, \\ s_1^2 A_j + s_2^2 B_j - \bar{s}_1^2 \bar{A}_j - \bar{s}_2^2 \bar{B}_j = -\dfrac{a_{16}}{a_{11}} \dfrac{P_{jx}}{2\pi hi} - \dfrac{a_{12}}{a_{11}} \dfrac{P_{jy}}{2\pi hi}, \\ \dfrac{A_j}{s_1} + \dfrac{B_j}{s_2} - \dfrac{\bar{A}_j}{\bar{s}_1} - \dfrac{\bar{B}_j}{\bar{s}_2} = \dfrac{a_{12}}{a_{22}} \dfrac{P_{jx}}{2\pi hi} + \dfrac{a_{26}}{a_{22}} \dfrac{P_{jy}}{2\pi hi}, \end{cases} \tag{4}$$

where a_{11}, a_{12}, a_{22}, a_{16} and a_{26} are elastic constants, s_1 and s_2 are two complex parameters of the plate material, P_{jx} and P_{jy} are two projections of the resultant vector P_j.

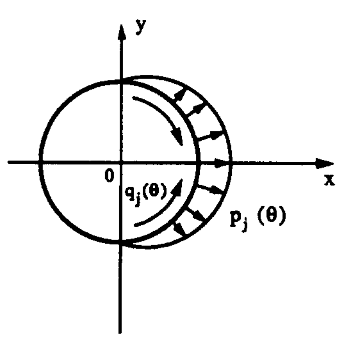

Fig. 4. The distribution of tangential stress along a hole contour.

Functions $[w_1(z_1-b_j)]^{-n}$ and $[w_2(z_2-b_j)]^{-n}$ $(j = 1,2,3,4;\ n = 1,2,3,\ldots)$ are holomorphic in the infinite region weakened by Hole j. Therefore, they are holomorphic in the interior of Hole m $(m \neq j)$ and continuous to its boundary of the complex z_1 and z_2 planes, respectively, and they can be expanded in Faber series (Fan and Wu, 1988; Xu and Fan, 1991):

$$\begin{cases} [w_1(z_1-b_j)]^{-n} = \sum_{k=0}^{\infty} A_{jnk} P_k(z_{1m}^*), \\ [w_2(z_2-b_j)]^{-n} = \sum_{k=0}^{\infty} B_{jnk} P_k(z_{2m}^*), \\ z_{1m}^* = z_1 - b_m \quad z_{2m}^* = z_2 - b_m, \end{cases} \tag{5}$$

where $P_0(z_{1m}^*) = P_0(z_{2m}^*) = 1$ and $P_k(z_{1m}^*)$, $P_k(z_{2m}^*)$ are kth Faber polynomials

$$P_k(z_{1m}^*) = [w_1(z_{1m}^*)]^k + t_1^k[w_1(z_{1m}^*)]^{-k},$$
$$P_k(z_{2m}^*) = [w_2(z_{2m}^*)]^k + t_2^k[w_2(z_{2m}^*)]^{-k} \tag{6}$$

$$(k = 1,2,3,\ldots), \tag{7}$$

$$t_1 = (1+\mathrm{i}s_1)/(1-\mathrm{i}s_1),$$
$$t_2 = (1+\mathrm{i}s_2)/(1-\mathrm{i}s_2).$$

Their boundary values

$$P_k(z_{1m}^*) = \sigma^k + t_1^k \sigma^{-k},$$
$$P_k(z_{2m}^*) = \sigma^k + t_2^k \sigma^{-k} \tag{8}$$

$$(k = 1,2,3,\ldots),$$

where

$$\sigma = \exp(\mathrm{i}\theta) = \cos\theta + \mathrm{i}\sin\theta$$

$$(0 \leqslant \theta < 2\pi) \tag{9}$$

is a point on the contour of the unit circle.

Functions $\ln(z_1-b_j)$ and $\ln(z_2-b_j)$ $(j = 1,2,3,4)$ are analytic but many-valued in the infinite region weakened by Hole j. However, they are holomorphic (analytic and one-valued) in the interior of Hole m $(m \neq j)$ and continuous to its boundary, of the complex z_1 and z_2 planes respectively, and they can be expanded in Faber series:

$$\ln(z_1-b_j) = \sum_{k=0}^{\infty} E_{jk} P_k(z_{1m}^*),$$
$$\ln(z_2-b_j) = \sum_{k=0}^{\infty} F_{jk} P_k(z_{2m}^*). \tag{10}$$

Coefficients A_{jnk}, B_{jnk}, E_{jk} and F_{jk} in the Faber series can be determined by the Fourier Expansion Method (Kosmodamianskii, 1981).

A partial sum up to the Nth power term is used instead of the respective series and the boundary values of two complex potential functions on the Hole m contour $(m = 1,2,3,4)$ can be expressed in the following forms:

$$\phi(z_1) = A_m \ln \sigma + \sum_{\substack{j=1 \\ j\neq m}}^{4} A_j\left[E_{jo} + \sum_{k=1}^{N} E_{jk}\left(\sigma^k + t_1^k \sigma^{-k}\right)\right] + \varphi_0 + \sum_{n=1}^{N} \phi_{mn}\sigma^{-n}$$
$$+ \sum_{\substack{j=1 \\ j\neq m}}^{4} \sum_{n=1}^{N} \varphi_{jn}\left[A_{jno} + \sum_{k=1}^{N} A_{jnk}(\sigma^k + t_1^k \sigma^{-k})\right],$$

$$\psi(z_2) = B_m \ln \sigma + \sum_{\substack{j=1 \\ j\neq m}}^{4} B_j\left[F_{jo} + \sum_{k=1}^{N} F_{jk}\left(\sigma^k + t_2^k \sigma^{-k}\right)\right] + \sum_{n=1}^{N} \psi_{mn}\sigma^{-n}$$
$$+ \sum_{\substack{j=1 \\ j\neq m}}^{4} \sum_{n=1}^{N} \psi_{jn}\left[B_{jno} + \sum_{k=1}^{N} B_{jnk}(\sigma^k + t_2^k \sigma^{-k})\right]. \quad (11)$$

The equilibrium condition is

$$P_1 + P_2 + P_3 + P_4 = P. \quad (12)$$

Assuming all of the fasteners are rigid, the continuity conditions can be easily obtained as follows:

(1) When the lower plate B is rigid, the continuity conditions are

$$u_1 = u_2 = u_3 = u_4, \quad (13)$$

where u_j ($j = 1, 2, 3, 4$) is the displacement of a point which is the intersection of a hole-centre line and the loading edge of the Hole j contour.

(2) When the lower plate is another orthotropic one, the conditions are

$$\begin{aligned} u_1 - u_2 &= u_1' - u_2', \\ u_2 - u_3 &= u_2' - u_3', \\ u_3 - u_4 &= u_3' - u_4', \end{aligned} \quad (14)$$

where u_j ($j = 1, 2, 3, 4$) are the displacements belonging to the upper plate A and u_j' belonging to the lower plate B.

(3) Especially when the upper and lower plates are both orthotropic with identical stiffnesses the symmetry conditions are

$$\begin{aligned} u_1 &= -u_4', \quad u_2 = -u_3', \\ u_3 &= -u_2', \quad u_4 = -u_1', \\ P_1 &= P_4, \quad P_2 = P_3. \end{aligned} \quad (15)$$

If we only consider the upper plate A, the conditions are

$$\begin{aligned} u_1 - u_2 &= u_3 - u_4, \\ P_1 &= P_4, \\ P_2 &= P_3. \end{aligned} \quad (16)$$

The boundary conditions on the Hole j contour ($j = 1, 2, 3, 4$) are

$$\phi(z_1)+\overline{\phi(z_1)}+\psi(z_2)+\overline{\psi(z_2)} = \int_0^s Y_n\, ds + c_{1j},$$

$$s_1\phi(z_1)+\overline{s_1\phi(z_1)}+s_2\psi(z_2)+\overline{s_2\psi(z_2)} = -\int_0^s X_n\, ds + c_{2j}, \tag{17}$$

where c_{1j} and c_{2j} are constants.

Substituting eqns (11) into eqns (17), the many-valued terms on both sides are equal and can be automatically cancelled out. By expanding the right-hand sides into complex Fourier series, and equating coefficients of the same power σ^m $(m = 0, \pm 1, \pm 2, \ldots, \pm N)$ on both sides, we can obtain $4(4N+2)$ linear equations. Combining equilibrium condition (12) and continuity condition (13) or (16), and assuming

$$c_{11} = c_{12} = 0 \tag{18}$$

we can obtain $4(4N+3)$ equations of φ_0, $\bar{\varphi}_0$, φ_{jn}, $\bar{\varphi}_{jn}$, ψ_{jn}, $\bar{\psi}_{jn}$, P_j $(j = 1,2,3,4;$ $n = 1,2,\ldots,M)$, c_{21}, c_{22}, c_{31}, c_{32}, c_{41}, c_{42}, and then evaluate the magnitude of P_j $(j = 1,2,3,4)$.

Until now, the above discussed joint assumed zero fitting tolerance. However, a positive fitting tolerance generally exists. The load distribution of a multi-fastener joint is significantly affected by the magnitudes and distribution of the fitting tolerances when all of the other parameters are fixed. To be on the safe side, we will only consider the serious case as shown in Fig. 5, where the first and fourth hole/pin connections are in contact, and the second and third have equal initial fitting tolerance δ_0. When the joint is subjected to an external load P monotonically increasing from zero to a certain (threshold) value P_0, the second and third hole/pin connections are not in contact with each other, and the external load is equally distributed by the first and fourth connections (Stage I). When $P > P_0$, all of the four connections are in contact simultaneously (Stage II), and a part of the load $(P-P_0)$, is distributed as in the former case, i.e. a joint with zero fitting tolerance. Then the whole load is distributed by the superposition theorem.

NUMERICAL EXAMPLES

(1) For comparison, we conducted the calculations for the four plates discussed by Wang and Han (1988). Plate a is isotropic (relatively rigid), of which elastic constants are assumed to be

$$E = 210{,}000 \text{ GPa (Young's modulus)},$$

$$v = 0.3 \text{ (Poisson's ratio)}.$$

Plates b, c and d (Table 1) with identical lamina stiffnesses:

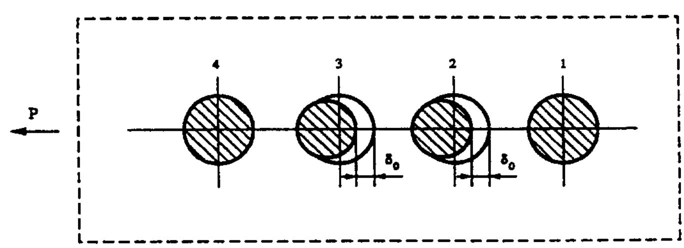

Fig. 5. The serious case of a joint existed a fitting tolerance.

Table 1. Ply orientations of plates

Plate	Percentage of ply orientations (%)		
	0-degree	±45-degree	90-degree
b $(\pm 45/0_8)_s$	80	20	0
c $(\pm 45/90_3/0/90_3/0)_s$	20	20	60
d $(\pm 45/90_8)_s$	0	20	80

$$E_1 = 160\ \text{GPa},\quad E_2 = 11.1\ \text{GPa},$$
$$G_{12} = 6.5\ \text{GPa},\quad \nu_{12} = 0.38.$$

We assume that

1. There is no fitting clearance.
2. There is no friction force.
3. The fasteners are rigid.
4. The plates are infinite in length and width.
5. The centre-to-centre distance is $L = 30$ mm, the hole diameter is $D = 5$ mm and the plate thickness $h = 2.5$ mm.

One of the outstanding advantages of the Faber series expansion is that the series converges very quickly. In our calculations, we take the partial sum up to $N = 10$ terms instead of using the infinite series (Fan and Wu, 1988).

Two numerical results, Fig. 6 in this paper and Fig. 6(a) in Wang and Han (1988), present a good agreement, where "unevenness factor" is defined as

$$f_i = P_i/\bar{P},\quad \bar{P} = P/4.$$

The same trend is that when axial stiffness of the plates decreases the load distribution tends to be more uneven. The relative differences between two numerical results are not more than 7.9%, although Wang and Han (1988) neglect the effect of the hole diameter.

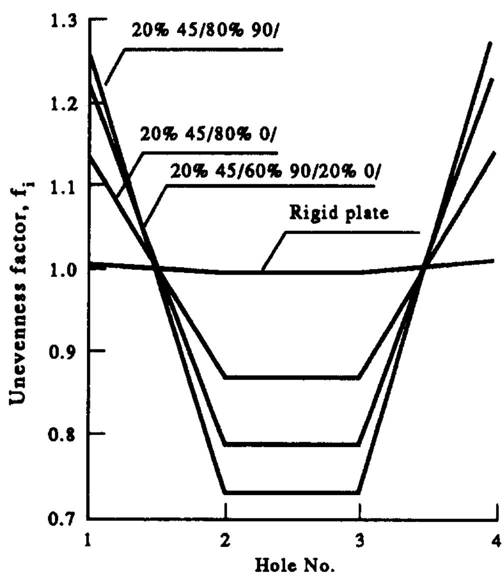

Fig. 6. The effect of lay-up.

(2) In order to check the effects of fitting tolerances, we calculate an equal stiffness joint shown in Figs 2 and 5. The upper and lower laminates $(0_4/\pm45/90_4)_s$ are carbon fiber reinforced T300/648 with lamina stiffness

$$E_1 = 98\text{ GPa}, \quad E_2 = 9.8\text{ GPa},$$

$$G_{12} = 5.2\text{ GPa}, \quad \nu_{12} = 0.31.$$

The geometric parameters

$$D = 5\text{ mm}, \quad h = 2.5\text{ mm}, \quad L = 30\text{ mm}.$$

The external loads are assumed to be 10, 20, 30, 40 and 50 kN, respectively.

When the initial fitting tolerance $\delta_0 = 0.03$ mm, the threshold load $P_0 = 6.70$ kN and the load distribution is shown in Fig. 7.

According to linear theory, when the initial fitting tolerance increases from $\delta_0 = 0.03$ mm to $\delta_0 = 0.06$ mm, the threshold load increases from $P_0 = 6.70$ kN to $P_0 = 13.4$ kN with the same proportion. At $P = 10$ kN, $P < P_0$, the second and third hole/pin connections are not in contact with each other, and the external load is distributed equally by the first and fourth connections (Stage I). At the higher loads, $P > P_0$, the load distribution is shown in Fig. 8.

It is shown from Figs 7 and 8 that when the initial fitting tolerance is fixed, along with an increase in the external load P, the load distribution of a multi-fastener joint tends to be more even.

Figure 9 presents the effect of fitting tolerance on the load distribution when the external load is fixed, $P = 30$ kN. The effect is important. The larger the fitting tolerance, the more uneven is the load distribution.

(3) In order to check effects of the friction between hole/pin connections, we consider three laminates, with the same structure parameters and lamina stiffness as above [Section (2)], but having zero fitting tolerance. The ply orientations of them are listed in Table 2. It is shown from the numerical results (Fig. 10) that at lower values of the friction coefficient

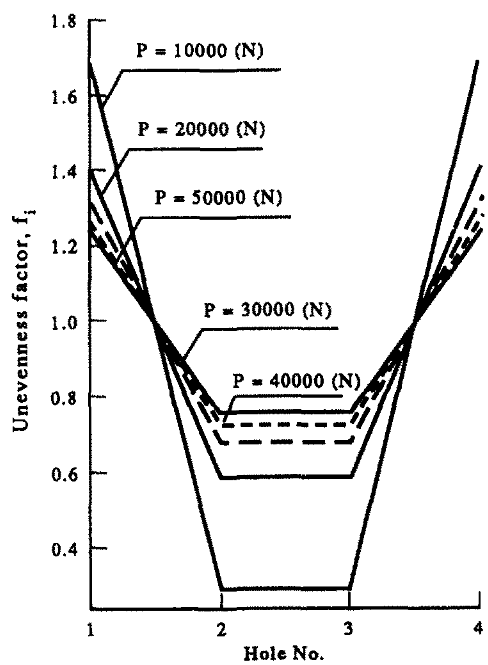

Fig. 7. The effect of fitting tolerance $\delta_0 = 0.03$ mm.

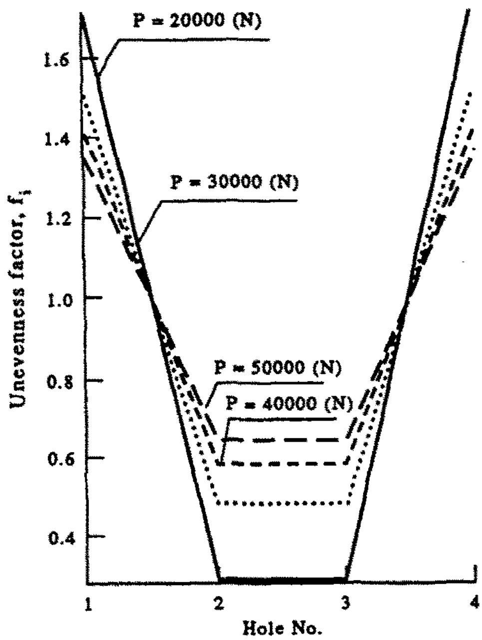

Fig. 8. The effect of fitting tolerance $\delta_0 = 0.06$ mm.

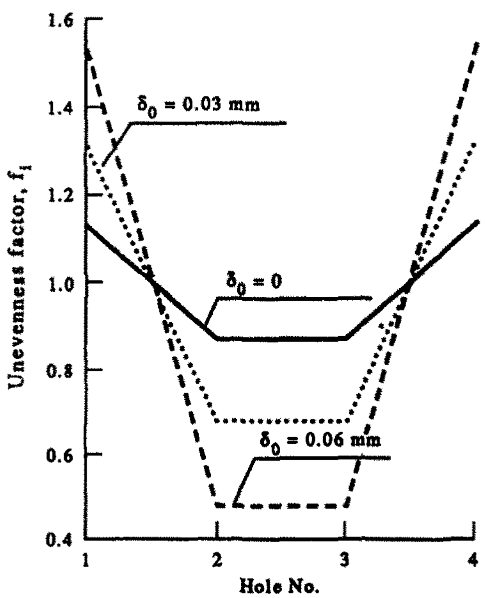

Fig. 9. The effect of different fitting tolerances at $P = 30$ kN.

Table 2. Ply orientations of laminates

	Percentage of ply orientations (%)		
Laminate No.	0-degree	±45-degree	90-degree
1	40	40	20
2	40	20	40
3	0	100	0

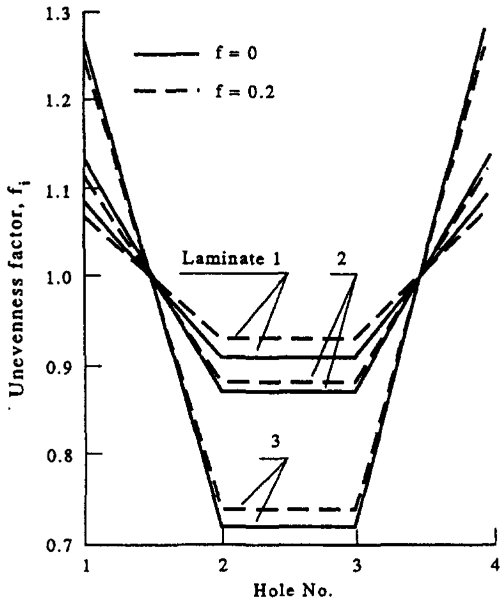

Fig. 10. The effect of friction.

(e.g. $f = 0.2$), its effect on the load distribution is not too large in magnitude, and the load distribution tends to become more even due to the existence of friction.

CONCLUSIONS

From the numerical results with 4-fastener laminated composite joints the following conclusions can be obtained:

(1) The effect of fitting tolerance on the load distribution of a multi-fastener joint is important. The larger the fitting tolerance, the more obvious is this effect.

(2) When the initial fitting tolerance is fixed, along with the increase in magnitude of the external load P, the load distribution of a multi-fastener joint tends to become more uniform among the individual fasteners.

(3) At lower values of the friction coefficient (e.g. $f = 0.2$), its effect on the load distribution is not too large in magnitude, and the load distribution tends to become more even due to the existence of friction.

(4) When the friction coefficient is fixed, along with the increasing of axial stiffness of the laminates, the load distribution tends to become more uniform. In a limiting case, as the jointed plates approach rigidity, all of the relative displacements between hole-centres approach zero and the external load is uniformly distributed among the hole/pin connections.

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